EXCEL COMPUTATIONAL DESIGN TOOL:
MULTIFUNCTIONAL STRUCTURE-BATTERY MATERIALS

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Table of Contents

Summary

This report describes a computational ***Structure-Battery Design Tool*** (SBDT) developed at the Naval Research Laboratory for analyzing the mechanical and electrical performance of multifunctional structure-battery materials configured in prismatic beam geometries. SBDT is implemented in Excel spreadsheet form and is capable of analyzing composite designs with several cross-section geometries including: circular-annular, rectangular-annular, and arbitrary-box. Instructions for using the SBDT and an overview of the calculations performed therein are included below.

Introduction

Multifunctional materials are “material systems” with more than one primary function. Multifunctional materials are developed to achieve system-level performance enhancements; local subsystem performance is a secondary concern. For example, the best structure-power multifunctional material may not be the strongest or stiffest, or have the largest energy storage capacity. What it should achieve is better system-level performance (e.g., flight time for an unmanned air vehicle) than can be obtained with any combination of unifunctional structure and power-storage materials.

The Structure-Battery Design Tool (SBDT) is implemented in the form of Excel spreadsheets and is capable of analyzing the mechanical and electrical energy storage properties for prismatic composites (Figure 1) with circular-annular, rectangular-annular, or arbitrary-box cross-section configurations (Figures 2). Inputs to the code include information related to geometry, configuration, material properties, etc., and outputs include mechanical stiffness and strength under axial, bending, torsion, shear, and buckling loads, electrical storage capacity, weight and volume based energy and power densities, etc. (Figure 3). Up to five distinct materials can be considered with the circular- and rectangular-annular cross-sections (Figure 2a and 2b) and seven distinct materials with the arbitrary-box cross-section (Figure 2c). The analysis of mechanical performance is based on classical Mechanics of Materials beam equations suitably extended for composites through the use of “modulus-weighted” cross-section properties in the stress and deformation equations. Nominal electrical performance is calculated assuming constant voltage and currents such that complete battery discharge occurs in one hour (i.e., 1C discharge rate).

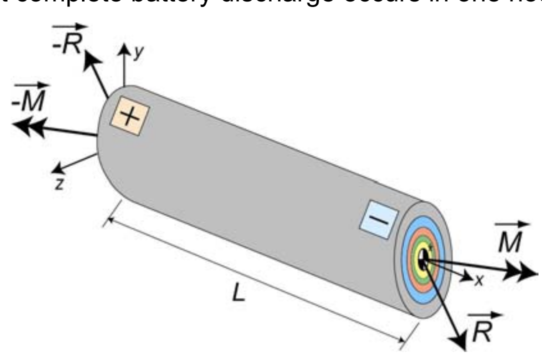
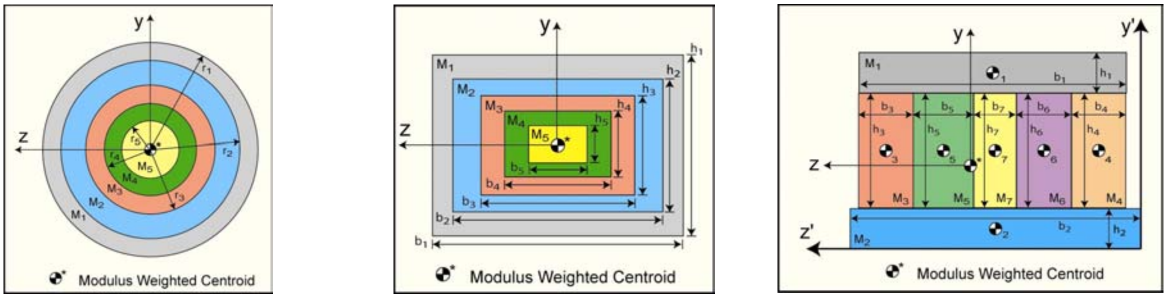


Figure 1: Schematic of a prismatic structure-battery material with internal load vectors $\vec{R}$ and $\vec{M}$ (defined on page 6, below) applied at the ends.



(a) **(b)** **(c)**

Figure 2: The three composite cross-section configurations: **(a)** circular-annular, **(b)** rectangular-annular, and **(c)** arbitrary-box. The *y-z* axes are located at the modulus-weighted cross-section centroid; the *y'-z'* axes are arbitrarily located.

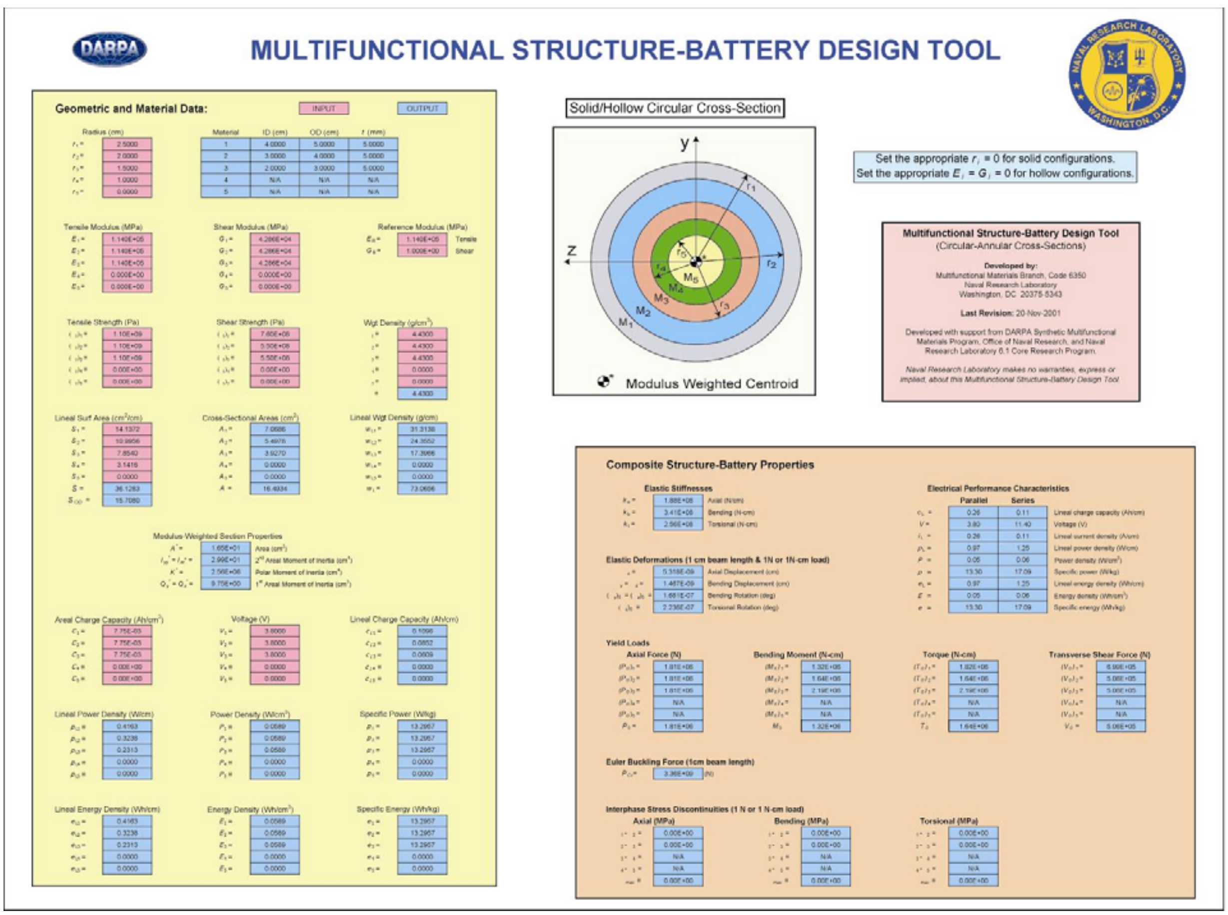


Figure 3: First page view of the SBDT for circular-annular cross-sections. The rose colored boxes signify input and the blue colored boxes are output. The yellow box on the left contains Geometric and Material Data input and calculations, and the tan box on the right contains Structure-Power property calculations for the composite beam.

The type of geometric and material input that is required by each template is essentially the same. The geometric input defines each material's geometry such as width, height, placement, surface area, etc. The material input is comprised of mechanical properties such as elastic stiffness, strength, density, and electrical properties, i.e., electric charge storage capacities and nominal voltages. The combination of the geometric and material input is employed to calculate geometric and modulus-weighted parameters such as centroids, cross-sectional areas, areal moments of inertia; mechanical parameters like stiffness, maximum displacements, yield strength, interfacial stresses; and electrical parameters for parallel and series connections like: voltage, power and energy densities, etc. Each SBDT has two calculation pages. The first page, denoted as "Input-Output", performs calculations on one particular design. The layout of this page is shown in Figure 3. The second page, denoted "Parametric", has all of inputs and outputs from "Input-Output" (page one) arranged horizontally in a single row. This provides an easy method for performing parametric studies on any of the input parameters.

Directions for Use

Colors are employed in the structure-battery design tool to distinguish input from output, and geometric and material data from structure-power material parameters. The color scheme is consistent in all design templates.

The first page in each template is named "Input-Output" and contains three boxes or windows. The yellow box on the left is Geometric and Material Data input and calculations, and the tan box on the right is Structure-Power properties for the composite. There is a green box below the other two that is not shown in Figure 3; it contains auxiliary data employed in calculating output in the other two boxes. Input cells are rose-colored and output cells are pale blue. The second page is named "Parametric" and uses the same color scheme as the first page.

The dimensional units used for each input and output cell are given. ***Note that all Inputs must conform to these units.*** Explanatory comments are also provided at various locations in the template.

Input-Output: Geometric and Material Data

The circular-annular design tool is denoted as C-SBDT, the rectangular-annular design tool is denoted as R-SBDT, and the arbitrary-box design tool is denoted as AB-SBDT. The first inputs in the yellow Geometric and Material Data box are: "Radius" for C-SBDT; "Width" and "Height" for R-SBDT; and "Width", "Height", and the $\hat{z}'$ centroid locations for the i^{th} material regions (i.e., $\bar{z}'_i$ values). As the input is entered, the blue columns to the right will be updated to show information on inner and outer diameters, wall thicknesses, and/or $\hat{y}'$ centroid locations for the i^{th} material region (i.e., $\bar{y}'_i$ values). Specific data entry instructions for solid and hollow configurations are given next to the cross-section on Page 1.

The next input consists of tensile and shear moduli, E_i and G_i, for each component material. The shear modulus value is automatically calculated using the isotropic material expression: $G_i = E_i / 2(1+\upsilon)$ with υ (Poisson ratio) equal to 0.33. The user can replace the calculated shear modulus value with actual data if so desired.

The next input is the reference moduli: E_R and G_R. Numerical values can be arbitrarily assigned. A value of 1.0 is commonly used or the moduli values for one of the component materials.

Yield strengths in tension $(\sigma_o)_i$ and shear $(\tau_o)_i$ follow. The shear strengths are automatically calculated using the Tresca criterion: $(\tau_o)_i = (\sigma_o)_i/2$. The user can replace these calculated shear strength values with actual data if so desired.

Component weight densities, ρ_i, are next. The last box in this column is output: the overall weight density of the composite.

The next input is surface area per unit beam length (i.e., lineal surface area), S_i. Each S_i entry is currently assigned the value of the average circumference or perimeter of the material region (e.g., $\pi(r_i + r_{i+1})$ for circular cross-sections). The user can replace these calculated values with actual data if so desired. The last two cells in the column are the total sum of lineal surface areas, S, and the lineal surface area of the outer layer, S_{OD}. Surface area quantities are used for electrical performance calculations on material regions comprised of multiple layers (e.g., multiple battery material layers).

The next two blocks are output. The first consists of the cross-sectional area of each material component, A_i, and total material cross-sectional area of the composite, A. The second block gives the material component weight per unit beam length (i.e., lineal weight densities), $w_{L\,i}$, and the total beam weight per unit beam length, w_L.

Modulus-weighted (MW) section properties are denoted by the " * " superscript. This includes parameters like: material area, A^*; cross-section location of the modulus weighted centroid, $\bar{y}'^*$ and $\bar{z}'^*$; 2nd moments of area, I^*_{yy}, I^*_{zz}, and I^*_{yz}; the two principal 2nd moments of area, $I^*_{1,2}$, and the corresponding angle between the $\hat{1}$-$\hat{2}$ principal axes relative to the $\hat{y}$-$\hat{z}$ MW centroidal axes, θ_p; polar moment of inertia, K^*; and 1st areal moments of inertia, Q^*_y and Q^*_z about the MW cross-section axes.

The last two input blocks are electrical data: electrical charge storage capacity per unit area, C_i, and nominal voltage, V_i, for each material component. The remaining blocks are electrical property outputs calculated for the i^{th} material regions including: electrical charge storage capacity per unit length (lineal capacity), $c_{L\,i}$; power capacity per unit length (lineal power density), $p_{L\,i}$; power per unit volume (power density), P_i; power per unit weight (specific power), p_i; electrical energy per unit length (lineal energy density), $e_{L\,i}$; electrical energy per unit volume (energy density), E_i; and electrical energy per unit weight (specific energy), e_i. All of the electrical property calculations assume a 1*C* current draw rate (i.e., current, in amperes, numerically equal to the total charge capacity in ampere-hours [Linden and Reddy, 2001]).

Structure-Power Material Parameters

The first block in the tan colored box contains elastic stiffness estimates for the composite beam. The entries in C-SBDT include axial, bending, and torsional stiffnesses. Bending stiffness values about the $\hat{y}$ and $\hat{z}$ axes are included with R-SBDT. Bending stiffnesses about $\hat{y}$, $\hat{z}$, and the two principal axes are included with AB-SBDT. Note that the modulus-weighted $\hat{y}$ - $\hat{z}$ axes are principal for the circular and rectangular cross-sections.

The block below gives elastic deformations for beams of unit length with unit applied loads. The entries in C-SBDT include axial and bending displacements, and bending and torsional rotations. R-SBDT includes bending displacements and rotations about the $\hat{y}$ and $\hat{z}$ axes, and AB-SBDT includes bending displacements and rotations about the $\hat{y}$, $\hat{z}$, and two principal axes.

The block to the right gives the electrical performance characteristics for the composite beam in the parallel and serial configurations. This includes quantities like: voltage, lineal and areal charge storage capacities, and lineal, volumetric, and weight-based energy and power densities.

Yield data are given in the next row of blocks. This includes yield values for axial force, bending moment, torque, and transverse shear for each material and for the composite as a whole. R-SBDT includes two bending moment and transverse shear yield loads corresponding to $\hat{y}$ and $\hat{z}$ loading. AB-SBDT includes four bending moment yield loads corresponding to $\hat{y}$, $\hat{z}$, and principal axis bending. The last entry in each block is the composite beam yield load, and it is taken as the minimum of all component material yield loads.

The next "block" gives the Euler buckling forces for composite beams of unit length. Values are computed for buckling about the principal 2nd moment of area axes.

The last row of blocks displays the normal and shear stress discontinuities at the interface between component materials with unit internal load values (i.e., axial force, bending moment(s) and torque). The last entry is the maximum stress discontinuity value for that particular loading.

Parametric Page

The entries on the Parametric page are exactly the same as those on the Input-Output page except for their physical layout. The cells on the Parametric page are arranged in a single horizontal row of the spreadsheet. Parametric studies can be performed by copying a "filled-in" row (i.e., inputs and subsequent outputs) to one or more empty rows followed by modifications to one or more of the column parameters. After performing the copy-paste and parameter modification steps, copy the entire page and then use the "Paste-Values Only" command to copy the only the numerical values onto another page for use in plotting. The "Paste-Values Only" command allows one to delete useless columns without affecting the other numbers. On the other hand, if changes are made on the Parametric page, then the page-copy-for-plotting steps will have to be re-performed.

Mechanical Analysis of Prismatic Structure-Battery Materials

The use of modulus-weighted cross-section properties provides a simple method for taking into account the composite nature of prismatic structure-battery materials in the analysis of mechanical performance [Allen and Haisler, 1985]. Standard “Mechanics of Materials” beam equation forms are retained with certain cross-sectional parameters replaced by modulus-weighted quantities.

The materials and cross-section are assumed to be constant along the length of the member (prismatic beam assumption), taken as the *x*-axis. Six types of internal loads are possible. They are assumed to be constant along the length (i.e., no *x* dependence) and to pass or act through the cross-section’s modulus-weighted centroid. They are represented, in Figure 1, by the vectors $\vec{R}$ and $\vec{M}$:

$$\vec{R} = P\hat{x} + V_y\hat{y} + V_z\hat{z} \tag{1}$$

$$\vec{M} = T\hat{x} + M_{xy}\hat{y} + M_{xz}\hat{z} \tag{2}$$

P is the axial force; V_y and V_z are the $\hat{y}$ and $\hat{z}$ transverse shear forces, respectively; T is the torque; and M_{xy} and M_{xz} are the $\hat{y}$ and $\hat{z}$ bending moments, respectively. The internal load sign convention is as follows:

Positive internal loads: positive face and positive coordinate directions or negative face and negative coordinate directions.

Negative internal loads: positive face and negative coordinate directions or negative face and positive coordinate directions.

Modulus-Weighted Cross-Section Property Calculations

The modulus-weighted cross-section properties, for discrete material distributions, are calculated as shown below [Allen and Haisler, 1985]. The $\hat{y}'$-$\hat{z}'$ coordinate axes can be arbitrarily located in the plane of the cross-section. The primary coordinate system is the $\hat{y}$-$\hat{z}$ modulus-weighted centroidal axes.

Modulus-Weighted Area:

$$A^* := \sum_{i=1}^{n} \frac{E_i}{E_\mathrm{R}} A_i \tag{3}$$

Modulus-Weighted Centroid Location:

$$\bar{y}'^* := \frac{1}{A^*}\sum_{i=1}^{n} \frac{E_i}{E_\mathrm{R}} A_i \bar{y}'_i \quad \text{and} \quad \bar{z}'^* := \frac{1}{A^*}\sum_{i=1}^{n} \frac{E_i}{E_\mathrm{R}} A_i \bar{z}'_i \tag{4}$$

Modulus-Weighted 2nd Area Moments of Inertia:

$$I_{yy}^* := \sum_{i=1}^{n} \frac{E_i}{E_\mathrm{R}} \left\{ \left(I_{yy0}\right)_i + \bar{z}_i^2 A_i \right\},\ I_{zz}^* := \sum_{i=1}^{n} \frac{E_i}{E_\mathrm{R}} \left\{ \left(I_{zz0}\right)_i + \bar{y}_i^2 A_i \right\},\ I_{yz}^* := \sum_{i=1}^{n} \frac{E_i}{E_\mathrm{R}} \left\{ \left(I_{yz0}\right)_i + \bar{y}_i \bar{z}_i A_i \right\} \tag{5}$$

$$I_{1,2}^* := \frac{I_{yy}^* + I_{zz}^*}{2} \pm \sqrt{\left(\frac{I_{yy}^* - I_{zz}^*}{2}\right)^2 + \left(I_{yz}^*\right)^2} \quad \text{and} \quad 2\theta_p = \tan^{-1}\left(\frac{2I_{yz}^*}{I_{yy}^* - I_{zz}^*}\right) \tag{6}$$

Modulus-Weighted Polar Moment of Inertia (parallel torsional springs):

$$K^* := \sum_{i=1}^{n} \frac{G_i}{G_\mathrm{R}} \left(K_0\right)_i \tag{7}$$

In Equations (3)-(7), n is the number of materials in the cross-section. E_i and G_i are the tensile and shear modulus for the i^{th} material. A_i is the cross-sectional area of the i^{th} material. $\bar{y}'_i$ and $\bar{z}'_i$ are the coordinates of the i^{th} material centroid relative to the $\hat{y}'$-$\hat{z}'$ axes. $\left(I_{yy0}\right)_i$, $\left(I_{zz0}\right)_i$, and $\left(I_{yz0}\right)_i$ are the second moments of area of the i^{th} material about the i^{th} material centroidal axes. $\bar{y}_i$ and $\bar{z}_i$ are the coordinates of the i^{th} material centroid relative to the modulus-weighted $\hat{y}$-$\hat{z}$ centroidal axes. $I^*_{1,2}$ are the principal 2nd moments of area and θ_p is the principal orientation angle relative to the $\hat{y}$-$\hat{z}$ centroidal axes. $\left(K_0\right)_i$ is the "effective" polar moment of inertia of the i^{th} material area relative to the i^{th} material centroidal axes. Expressions for $\left(K_0\right)_i$ pertaining to rectangular cross-sections are given below. Finally, E_R and G_R are the reference tensile and shear moduli; their value is arbitrary and any convenient value can be selected for use.

Note that $I^*_{yz}=0$ for the circular- and rectangular-annular cross-sections, for doubly symmetric cross-sections, and for cross-sections with the modulus-weighted coordinate axes in the principal orientation. Great simplifications occur in the stress and deformation expressions when this cross-product term is zero.

Stress Calculations

The normal stress in the i^{th} material due to internal axial force and/or bending moment is given by:

$$\left(\sigma_x\right)_i=\frac{E_i}{E_{\mathrm{R}}}\left\{\frac{P}{A^*}-\frac{M_{xz}I^*_{yy}+M_{xy}I^*_{yz}}{\left(I^*_{yy}I^*_{zz}-I^{*\,2}_{yz}\right)}y+\frac{M_{xy}I^*_{zz}+M_{xz}I^*_{yz}}{\left(I^*_{yy}I^*_{zz}-I^{*\,2}_{yz}\right)}z\right\} \tag{8}$$

where "*y*" and "*z*" are positive or negative distances ***from*** the modulus weighted centroid ***to*** the point of interest.

The shear stress in the i^{th} material due to an internal torque is given by:

$$\tau_i=\frac{G_i}{G_{\mathrm{R}}}\frac{T\times r}{K^*} \tag{9}$$

for circular-annular cross-sections where "*r*" is the distance ***from*** the modulus weighted centroid ***to*** the radial point of interest. The maximum torsional shear stress for solid rectangular sections is given by:

$$\tau_{\max}=\frac{G_{solid}}{G_R}\frac{K_{solid}}{K^*}\frac{T\left(3b+1.8h\right)}{b^2h^2} \tag{10}$$

It occurs on the perimeter at the midpoint of the longest side. The geometry is shown in Figure 4a with the "*b*" dimension denoting the longer side.

The torsional shear stress for hollow rectangular sections is given by:

$$\left(\tau_i\right)_{b,h}=\frac{G_i}{G_R}\frac{\left(K_0\right)_i}{K^*}\frac{T}{2t_{b,h}\left(b-t_b\right)\left(h-t_h\right)} \tag{11}$$

and is assumed constant through the wall (thin-walled assumption). The geometry is shown in Figure 4b.

Expressions for the "effective" polar moment of inertias are given by:

$$K_{solid} = bh^3\left[\frac{1}{3} - 0.21\frac{h}{b}\left(1 - \frac{h^4}{12b^4}\right)\right] \tag{12}$$

$$K_0 = \frac{2t_h t_b (b - t_b)^2 (h - t_h)^2}{bt_b + ht_h - t_b{}^2 - t_h{}^2} \tag{13}$$

Eqs. (10) and (12) are obtained from [Roark and Young, 1975] and Eqs. (11) and (13) from torsion analysis of arbitrary shaped thin-walled tubes [Allen and Haisler, 1985].

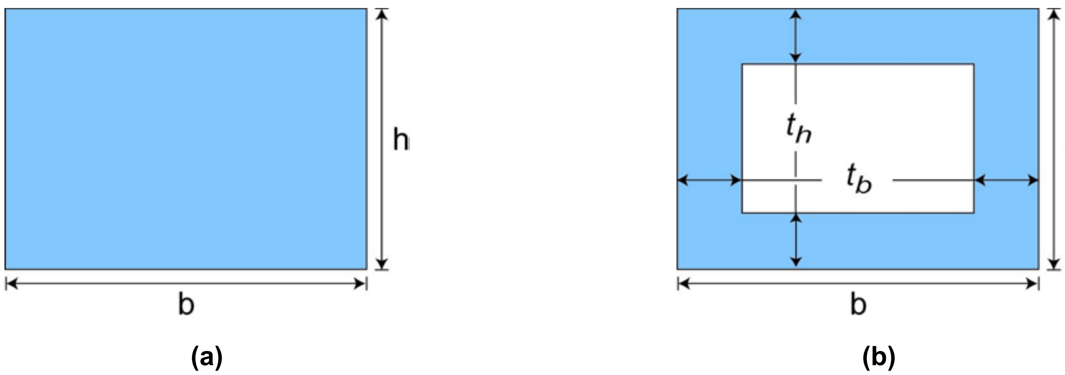


Figure 4: Solid **(a)** and hollow **(b)** rectangular cross-section geometries and variables.

Shear stresses, τ_{xy} and τ_{xz}, due to transverse shear forces are given by:

$$\tau_{xy} = \frac{\left(-V_y I^*_{yy} + V_z I^*_{yz}\right)}{\left(I^*_{yy} I^*_{zz} - I^{*\,2}_{yz}\right)}\frac{Q^*_z}{t_z} \quad \text{and} \quad \tau_{xz} = \frac{\left(-V_z I^*_{zz} + V_y I^*_{yz}\right)}{\left(I^*_{yy} I^*_{zz} - I^{*\,2}_{yz}\right)}\frac{Q^*_y}{t_y} \tag{14}$$

where Q^*_y and Q^*_z are the modulus-weighted first moments of area about $\hat{y}$-$\hat{z}$ axes:

$$Q^*_y := \sum_{i=1}^{n} \frac{E_i}{E_R}\left(Q_y\right)_i \quad \text{and} \quad Q^*_z := \sum_{i=1}^{n} \frac{E_i}{E_R}\left(Q_z\right)_i . \tag{15}$$

with $\left(Q_y\right)_i := \bar{z}_i A_i$ and $\left(Q_z\right)_i := \bar{y}_i A_i$, the first moments of area of the i^{th} material about the modulus-weighted centroidal axes (i.e., $\hat{y}$ and $\hat{z}$). Also, t_y and t_z are the material thickness (total) along $\hat{y}$ or $\hat{z}$ cross-section "cuts" where the shear stresses are computed.

Yield Failure Loads

Failure by yielding is assumed to occur when any one of the component materials yields. Internal load values, ***considered one at a time***, for which yielding occurs are calculated using the stress equations, Eqs. (8)-(10), (11), and (14), and tensile and shear yield strength input values. The equations are detailed below:

ALL: $$\left(P_0\right)_i = \frac{E_{\mathrm{R}}}{E_i} A^*\left(\sigma_0\right)_i \quad \text{and} \quad P_0 := \min_i\left(P_0\right)_i \tag{16}$$

C-SBDT: $$\left(M_0\right)_i = \frac{E_{\mathrm{R}}}{E_i} \frac{I^*}{\left(r_{\max}\right)_i}\left(\sigma_0\right)_i \quad \text{and} \quad M_0 := \min_i\left(M_0\right)_i \tag{17}$$

$$\left(T_0\right)_i = \frac{G_{\mathrm{R}}}{G_i} \frac{K^*}{\left(r_{\max}\right)_i}\left(\tau_0\right)_i \quad \text{and} \quad T_0 := \min_i\left(T_0\right)_i \tag{18}$$

$$\left(V_0\right)_i = \frac{t_z\, I_{zz}^*}{Q_z^*}\left(\tau_0\right)_i \quad \text{and} \quad V_0 := \min_i\left(V_0\right)_i \tag{19}$$

R-SBDT: $$\left(M_{xy0}\right)_i = \frac{E_{\mathrm{R}}}{E_i} \frac{I_{yy}^*}{\left(z_{\max}\right)_i}\left(\sigma_0\right)_i \quad \text{and} \quad M_{xy0} := \min_i\left(M_{xy0}\right)_i \tag{20}$$

$$\left(M_{xz0}\right)_i = \frac{E_{\mathrm{R}}}{E_i} \frac{I_{zz}^*}{\left(y_{\max}\right)_i}\left(\sigma_0\right)_i \quad \text{and} \quad M_{xz0} := \min_i\left(M_{xz0}\right)_i \tag{21}$$

$$\left(T_0\right)_{solid} = \frac{G_R}{G_{solid}} \frac{K^*}{K_{solid}} \frac{b^2 h^2}{\left(3b + 1.8h\right)}\left(\tau_0\right)_{solid} \quad , \quad \left(T_0\right)_i = \frac{G_R}{G_i} \frac{K^*}{\left(K_0\right)_i} 2t_{b,h}\left(b - t_b\right)\left(h - t_h\right)\left(\tau_0\right)_i \tag{22}$$

and $$T_0 := \min_i\left\{\left(T_0\right)_{solid}, \left(T_0\right)_i\right\} \tag{23}$$

$$\left(V_{y0}\right)_i = \frac{t_z\, I_{zz}^*}{Q_z^*}\left(\tau_0\right)_i \quad \text{and} \quad V_{y0} := \min_i\left(V_{y0}\right)_i \tag{24}$$

$$\left(V_{z0}\right)_i = \frac{t_y\, I_{yy}^*}{Q_y^*}\left(\tau_0\right)_i \quad \text{and} \quad V_{z0} := \min_i\left(V_{z0}\right)_i \tag{25}$$

AB-SBDT: $$\left(M_{xy0}\right)_i = \frac{E_{\mathrm{R}}}{E_i} \frac{\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)}{\left(\left(z_{\max}\right)_i I_{zz}^* - \left(y_{\max}\right)_i I_{yz}^*\right)}\left(\sigma_0\right)_i \quad \text{and} \quad M_{xy0} := \min_i\left(M_{xy0}\right)_i \tag{26}$$

$$\left(M_{xz0}\right)_i = \frac{E_{\mathrm{R}}}{E_i} \frac{\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)}{\left(\left(z_{\max}\right)_i I_{yz}^* - \left(y_{\max}\right)_i I_{yy}^*\right)}\left(\sigma_0\right)_i \quad \text{and} \quad M_{xz0} := \min_i\left(M_{xz0}\right)_i \tag{27}$$

$$\left(M_{10}\right)_i = \frac{E_{\mathrm{R}}}{E_i} \frac{I_1^*}{\left(d2_{\max}\right)_i}\left(\sigma_0\right)_i \quad \text{and} \quad M_{10} := \min_i\left(M_{10}\right)_i \tag{28}$$

$$\left(M_{20}\right)_i = \frac{E_{\mathrm{R}}}{E_i} \frac{I_2^*}{\left(d1_{\max}\right)_i}\left(\sigma_0\right)_i \quad \text{and} \quad M_{20} := \min_i\left(M_{20}\right)_i \tag{29}$$

$$\left(T_0\right)_i = \frac{G_R}{G_i} \frac{K^*}{K_i} \frac{b^2 h^2}{\left(3b + 1.8h\right)}\left(\tau_0\right)_i \quad \text{and} \quad T_0 := \min_i\left\{\left(T_0\right)_i\right\} \tag{30}$$

$$\left(V_{y0}\right)_i = \frac{t_z}{Q_z^*} \frac{\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)}{I_{yy}^*}\left(\tau_0\right)_i \quad \text{and} \quad V_{y0} := \min_i\left(V_{y0}\right)_i \tag{31}$$

$$\left(V_{z0}\right)_i = \frac{t_y}{Q_y^*} \frac{\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)}{I_{zz}^*}\left(\tau_0\right)_i \quad \text{and} \quad V_{z0} := \min_i\left(V_{z0}\right)_i \tag{32}$$

Buckling Failure Forces

Critical buckling forces for the principal planes (*xy* and *xz* planes for the circular and rectangular cross-sections) are calculated using the Euler buckling equation with pinned end conditions and unit beam length (i.e., $L = 1\,cm$):

$$\left(P_{Cr}\right)_1 = \frac{\pi^2 E_R I_1^*}{L^2} \quad \text{and} \quad \left(P_{Cr}\right)_2 = \frac{\pi^2 E_R I_2^*}{L^2} \tag{33}$$

The minimum buckling load, for a given cross-section configuration, corresponds with the minimum principal 2nd areal moment value.

Interphase Stress Discontinuities

The difference in the normal and the torsional shear stresses at the interface between the component materials is calculated for various unit internal loading cases.

Deformation Calculations

Displacements and rotations for the composite beams are calculated assuming that the internal loads are constant along the length (i.e., no *x* dependence):

$$\frac{du_o}{dx} = \frac{P}{E_R A^*} \quad \Rightarrow \quad u_o = \frac{PL}{E_R A^*} \tag{34}$$

$$\frac{d^2 v_o}{dx^2} = \frac{M_{xz} I_{yy}^* + M_{xy} I_{yz}^*}{E_R\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \quad \Rightarrow \quad v_o = \frac{\left(M_{xz} I_{yy}^* + M_{xy} I_{yz}^*\right)L^2}{2E_R\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \tag{35}$$

$$\frac{d^2 w_o}{dx^2} = -\frac{M_{xy} I_{zz}^* + M_{xz} I_{yz}^*}{E_R\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \quad \Rightarrow \quad w_o = -\frac{\left(M_{xy} I_{zz}^* + M_{xz} I_{yz}^*\right)L^2}{2E_R\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \tag{36}$$

$$\frac{d\theta_x}{dx} = \frac{T}{G_R K^*} \quad \Rightarrow \quad \theta_x = \frac{TL}{G_R K^*} \tag{37}$$

$$\frac{d\theta_y}{dx} = \frac{M_{xy} I_{zz}^* + M_{xz} I_{yz}^*}{E_R\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \quad \Rightarrow \quad \theta_y = \frac{\left(M_{xy} I_{zz}^* + M_{xz} I_{yz}^*\right)L}{E_R\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \tag{38}$$

$$\frac{d\theta_z}{dx} = \frac{M_{xz} I_{yy}^* + M_{xy} I_{yz}^*}{E_R\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \quad \Rightarrow \quad \theta_z = \frac{\left(M_{xz} I_{yy}^* + M_{xy} I_{yz}^*\right)L}{E_R\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \tag{39}$$

In the above equations, u_o, v_o, and w_o are the *x*, *y*, *z* displacements, and θ_x, θ_y, and θ_z are the *x*, *y* and *z*-axis rotations of the modulus-weighted centroid.

Elastic Stiffness Calculations

Elastic stiffness values for the various internal loads are calculated using Eqs. (34) - (39):

Axial:

$$P = k_a u_0 \quad \Rightarrow \quad k_a := \frac{E_R A^*}{L} \tag{40}$$

Torsion:
$$T = k_t\theta_x \quad \Rightarrow \quad k_t := \frac{G_{\mathrm{R}}K^*}{L} \tag{41}$$

Bending-Symmetric:
$$M_{xy} = (k_b)_{yy}\theta_y \quad \Rightarrow \quad (k_b)_{yy} := \frac{E_{\mathrm{R}}\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)}{I_{zz}^* L} \tag{42}$$

$$M_{xz} = (k_b)_{zz}\theta_z \quad \Rightarrow \quad (k_b)_{zz} := \frac{E_{\mathrm{R}}\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)}{I_{yy}^* L} \tag{43}$$

Antisymmetric: $M_{xy} = (k_b)_{yz}\theta_z$ and $M_{xz} = (k_b)_{zy}\theta_y \Rightarrow (k_b)_{yz} = (k_b)_{zy} := \dfrac{E_{\mathrm{R}}\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)}{I_{yz}^* L}$ (44)

The values listed in the Elastic Stiffnesses block are computed assuming unit length (i.e., $L = 1$ cm). Also, note that $(k_b)_{yy} = (k_b)_{zz}$ and $(k_b)_{yz} = (k_b)_{zy} = \infty$ for circular configurations and that $(k_b)_{yz} = (k_b)_{zy} = \infty$ for rectangular configurations.

Elastic Deflection Calculations

Elastic deformations are calculated for various internal loadings assuming unit load and unit length values (e.g., $P = 1\,N$ and $L = 1\,cm$):

$$u_0 = \frac{P}{k_a} \quad \Rightarrow \quad (u_0)_0 := \delta_a = \frac{1}{E_{\mathrm{R}}A^*} \tag{45}$$

$$v_0 = \frac{M_{xz}L}{2(k_b)_{zz}} + \frac{M_{xy}L}{2(k_b)_{yz}} \quad \Rightarrow \quad (\delta_y)_s = \frac{I_{yy}^*}{2E_{\mathrm{R}}\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \quad \text{and} \quad (\delta_y)_a = \frac{I_{yz}^*}{2E_{\mathrm{R}}\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \tag{46}$$

$$w_0 = \frac{-M_{xy}L}{2(k_b)_{yy}} + \frac{-M_{xz}L}{2(k_b)_{yz}} \quad \Rightarrow \quad (\delta_z)_s = \frac{-I_{zz}^*}{2E_{\mathrm{R}}\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \quad \text{and} \quad (\delta_z)_a = \frac{-I_{yz}^*}{2E_{\mathrm{R}}\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \tag{47}$$

$$\theta_x = \frac{T}{k_t} \quad \Rightarrow \quad (\theta_x)_0 = \frac{1}{G_{\mathrm{R}}K^*} \tag{48}$$

$$\theta_y = \frac{-M_{xy}}{(k_b)_{yy}} + \frac{-M_{xz}}{(k_b)_{yz}} \quad \Rightarrow \quad (\theta_y)_s = \frac{-I_{zz}^*}{E_{\mathrm{R}}\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \quad \text{and} \quad (\theta_y)_a = \frac{-I_{yz}^*}{E_{\mathrm{R}}\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \tag{49}$$

$$\theta_z = \frac{M_{xz}}{(k_b)_{zz}} + \frac{M_{xy}}{(k_b)_{yz}} \quad \Rightarrow \quad (\theta_z)_s = \frac{I_{yy}^*}{E_{\mathrm{R}}\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \quad \text{and} \quad (\theta_z)_a = \frac{I_{yz}^*}{E_{\mathrm{R}}\left(I_{yy}^* I_{zz}^* - I_{yz}^{*\,2}\right)} \tag{50}$$

Summarizing:

δ_a is the $\hat{x}$ deflection for $P = 1\,N$ and $L = 1\,cm$

δ_y or $(\delta_y)_s$ is the "symmetric" part of the $\hat{y}$ deflection for $M_{xz} = 1\,N\text{-}cm$ and $L = 1\,cm$

$(\delta_y)_a$ is the "antisymmetric" part of the $\hat{y}$ deflection for $M_{xy} = 1\,N\text{-}cm$ and $L = 1\,cm$

δ_z or $(\delta_z)_s$ is the "symmetric" part of the $\hat{z}$ deflection for $M_{xy} = 1\, N\text{-}cm$ and $L = 1\, cm$

$(\delta_z)_a$ is the "antisymmetric" part of the $\hat{z}$ deflection for $M_{xz} = 1\, N\text{-}cm$ and $L = 1\, cm$

θ_x is the *x*-axis rotation for $T = 1\, N\text{-}cm$ and $L = 1\, cm$

θ_y or $(\theta_y)_s$ is the "symmetric" part of the *y*-axis rotation for $M_{xy} = 1\, N\text{-}cm$ and $L = 1\, cm$

$(\theta_y)_a$ is the "antisymmetric" part of the *y*-axis rotation for $M_{xz} = 1\, N\text{-}cm$ and $L = 1\, cm$

θ_z or $(\theta_z)_s$ is the "symmetric" part of the *z*-axis rotation for $M_{xz} = 1\, N\text{-}cm$ and $L = 1\, cm$

$(\theta_z)_a$ is the "antisymmetric" part of the *z*-axis rotation for $M_{xy} = 1\, N\text{-}cm$ and $L = 1\, cm$

Cross-deflections and rotations (i.e., symmetric and antisymmetric parts) occur when $I^*_{yz} \neq 0$.

Weight Density Calculations

The weight density of the composite beam is defined as:

$$\rho := \frac{\sum_i \rho_i A_i}{A} \tag{51}$$

where ρ_i is the weight density of the i^{th} material, A_i is the cross-section area of the i^{th} material region, and A is total cross-section area. The weight density per unit length (lineal density), is defined as:

$$w_{\mathrm{L}} := \rho A \tag{52}$$

Battery Performance Calculations

The electrical charge storage capacity per unit area (areal charge capacity), C_i , and the nominal voltage, V_i , of the i^{th} material are supplied as input data in the yellow Geometric and Material Data box. Areal charge storage capacity is a common parameter used in rating the performance of battery cell materials supplied in thin-sheet form. The "theoretical" charge storage capacity, or the total amount of electrical charge transferred in complete reaction of active battery material, can be calculated knowing the amount of active material present in the cell and the chemical and electron transfer reactions involved.

Charge storage capacity per unit length (lineal charge capacity) is output in an adjacent block:

Lineal Charge Capacity:

$$c_{L\,i} := C_i S_i \tag{53}$$

S_i is the surface area of the i^{th} material per unit beam length (lineal surface area).

The next row of blocks gives the power capacity per unit length (lineal power density), the power per unit volume (power density), and the power per unit weight (specific power) for each material component. The calculated power quantities assume that discharge occurs at the nominal voltage and at the 1*C* current draw rate (i.e., current in *A* numerically equal to charge storage capacity in *A-h*) (Linden & Reddy, 2001). At the 1*C* discharge rate, the battery energy is completely drained in 1 hour.

Lineal Power Density: $$p_{Li} := \frac{c_i V_i}{1\,hr} \tag{54}$$

Power Density: $$P_i := \frac{p_{Li}}{A_i} \tag{55}$$

Specific Power: $$p_i := \frac{P_i}{\rho_i} \tag{56}$$

The last row of blocks gives the stored energy per unit length (lineal energy density), the stored energy per unit volume (energy density), and the stored energy per unit weight (specific energy) for each material component. Again, the calculated quantities assume that discharge occurs at the nominal voltage and at the 1 *C* current draw rate:

Lineal Energy Density: $$e_{Li} := p_{Li} \times 1\,hr \tag{57}$$

Energy Density: $$E_i := P_i \times 1\,hr \tag{58}$$

Specific Energy: $$e_i := p_i \times 1\,hr \tag{59}$$

Parameters in the ***Electrical Performance Characteristics*** block located in the tan-colored ***Composite Structure-Battery Properties*** box are calculated for both parallel and serial configurations and include: electrical charge storage capacity per unit length (lineal charge capacity), voltage, 1 *C* current capacity per unit length (lineal current density), power per unit length (lineal power density), power per unit enclosed (C- and R-SBDT) volume (power density), power per unit weight (specific power), stored energy per unit length (lineal energy density), energy per unit enclosed (C- and R-SBDT) volume (energy density), and energy per unit weight (specific energy) for the composite beam. The equations used in the calculations are detailed below:

	Parallel	**Series**	
Lineal Charge Capacity:	$c_L = \sum_i c_{Li}$	$c_L = c_{L1} = c_{L2} = c_{L3} = \cdots$	(60)
Voltage:	$V = V_1 = V_2 = V_3 = \cdots$	$V = \sum_i V_i$	(61)
Lineal 1*C* Current Density:	$i_L = \frac{c_L}{1\,hr}$	$i_L = \frac{c_L}{1\,hr}$	(62)
Lineal Power Density:	$p_L = \frac{c_L V}{1\,hr}$	$p_L = \frac{c_L V}{1\,hr}$	(63)
Power Density:	$P = \frac{p_L}{A_0}$	$P = \frac{p_L}{A_0}$	(64)
Specific Power:	$p = \frac{p_L}{w_L}$	$p = \frac{p_L}{w_L}$	(65)
Lineal Energy Density:	$e_L = c_L V$	$e_L = c_L V$	(66)
Energy Density:	$E = \frac{e_L}{A_0}$	$E = \frac{e_L}{A_0}$	(67)

	Parallel	Series	
Specific Energy:	$e = \dfrac{e_L}{w_L}$	$e = \dfrac{e_L}{w_L}$	(68)

The c_L (Series) quantity in Eq. (60) is assigned the first nonzero c_{Li} value and the V (Parallel) quantity in Eq. (61) is assigned the first nonzero V_i value. The quantity, A_0 , denotes the cross-section area defined by the outermost perimeter, for the circular and rectangular cross-sections, and total material area for arbitrary-box sections.

References

1. R. J. Roark and W. C. Young, “Formulas for Stress and Strain”, 5th Edition, McGraw-Hill, 1975.
2. D. H. Allen and W. E. Haisler, “Introduction to Aerospace Structural Analysis”, John Wiley, 1985.
3. D. Linden and T. B. Reddy, “Handbook of Batteries”, 3rd Edition, McGraw-Hill, 2001.

Nomenclature Table

A	Total material-filled cross-section area of the composite [cm^2]
A_i	Cross-sectional area of i^{th} material region [cm^2]
A_0	Projected cross-section area of the outer perimeter [cm^2]
A^*	Modulus-weighted cross-sectional area [cm^2]
b	Width of a rectangular solid component [cm]
$c_{L\,i}$	Charge storage capacity per unit length (lineal capacity) of i^{th} material region [Ah/cm]
c_{L}	Charge storage capacity per unit length (lineal charge capacity) of composite [Ah/cm]
C_i	Charge storage capacity per unit area (areal charge capacity) of i^{th} material [Ah/cm^2]
e	Stored energy per unit weight (specific energy) of composite [Wh/Kg]
e_i	Stored electrical energy per unit weight (specific energy) of i^{th} material region [Wh/Kg]
e_{L}	Stored energy per unit length (lineal energy density) of composite [Wh/cm]
$e_{L\,i}$	Stored electrical energy per unit length (lineal energy density) of i^{th} material [Wh/cm]
E	Stored energy per unit volume (energy density) of composite [Wh/cm^3]
E_i	Stored electrical energy per unit volume (energy density) of i^{th} material region [Wh/cm^3]
E_i , G_i	Tensile and shear modulus of i^{th} material region [MPa]
E_{R} , G_{R}	Reference tensile and shear modulus [MPa]
G_{solid}	Shear modulus of rectangular solid component [MPa]
h	Height of a rectangular solid component [cm]
i_L	1 *C* current capacity of the composite per unit length (lineal current capacity) [A/cm]
$\left(I_{yy0}\right)_i$, $\left(I_{zz0}\right)_i$, $\left(I_{yz0}\right)_i$	2^{nd} areal moments of the i^{th} material region about their centroidal axes [cm^4]
I_1^* , I_2^*	Principal modulus-weighted 2^{nd} moments of area [cm^4]
I_{yy}^* , I_{zz}^* , I_{yz}^*	Modulus-weighted 2^{nd} moments of area [cm^4]
k_a	Axial stiffness of composite [N/cm]
k_t	Torsional stiffness of composite [N-cm]
$\left(k_b\right)_{yy}$, $\left(k_b\right)_{zz}$	Symmetric bending stiffness components [N-cm]
$\left(k_b\right)_{yz} = \left(k_b\right)_{zy}$	Antisymmetric bending stiffness components [N-cm]
$\left(K_0\right)_i$	"Effective" polar moment of inertia of i^{th} material region about their centroidal axes [cm^4]
K_{solid}	"Effective" polar moment of a rectangular solid component [cm^4]
K^*	Modulus-weighted "effective" polar moment of inertia [cm^4]
M_{xy} , M_{xz}	Internal bending moments about *y* and z axes [N-cm]
M_{xz0} , M_{xy0}	Bending moment components that cause yielding in composite [N-cm]
$\left(M_{xz0}\right)_i$, $\left(M_{xy0}\right)_i$	Bending moment components that cause yielding in i^{th} material region [N-cm]
p	Power per unit weight (specific power) of composite [W/kg]
p_i	Power per unit weight (specific power) of i^{th} material region [W/kg]
p_{L}	Power per unit length (lineal power density) of composite [W/cm]

$p_{L\,i}$	Power capacity per unit length (lineal power density) of i^{th} material region [W/cm]
P	Power per unit volume (power density) of composite [W/cm^3]
P	Internal axial force [N]
P_i	Power per unit volume (power density) of i^{th} material region [W/cm^3]
P_0	Axial force that causes yield in composite [N]
$(P_0)_i$	Axial force that causes yield in i^{th} material region [N]
$(P_{Cr})_1$, $(P_{Cr})_2$	Euler buckling forces for bending in the principal planes [N]
$(Q_y)_i$, $(Q_z)_i$	1st areal moment of i^{th} material region [cm^3]
Q_y^*, Q_z^*	Modulus-weighted 1st areal moments [cm^3]
r	Radial coordinate *from* modulus weighted centroid *to* point of interest [cm]
S	Total sum of lineal surface areas [cm^2/cm]
S_i	Surface area per unit beam length (lineal surface area) of i^{th} material region [cm^2/cm]
S_{OD}	Surface area of the outer layer [cm^2/cm]
t_b	Thickness of rectangular cross-section of i^{th} material region in width direction [cm]
t_h	Thickness of rectangular cross-section of i^{th} material region in height direction [cm]
t_y, t_z	Material thickness in the $\hat{y}$ and $\hat{z}$ directions [cm]
T	Internal torque [N-cm]
T_0	Torque that causes yield in composite [N-cm]
$(T_0)_i$	Torque that causes yield in i^{th} material region [N-cm]
u_o, v_o, w_o	Displacement components of the MW centroid [cm]
V	Nominal voltage of the composite [Volts]
V_i	Nominal voltage of i^{th} material region [Volts]
V_y, V_z	Internal transverse shear forces in *y* and z directions [N]
V_{y0}, V_{z0}	Transverse shear force components that cause yielding in composite [N]
$(V_{y0})_i$, $(V_{z0})_i$	Transverse shear force components that cause yielding in i^{th} material region [N]
w_L	Weight per unit length (lineal weight density) of composite [g/cm]
$w_{L\,i}$	Weight per unit beam length (lineal weight densities) of i^{th} material region [g/cm]
$\hat{x}$, $\hat{y}$, $\hat{z}$	Unit vectors in the modulus weighted centroidal coordinate system
$\hat{x}'$, $\hat{y}'$, $\hat{z}'$	Units vectors in an arbitrary coordinate system
y, z	Coordinates *from* modulus weighted centroid *to* point of interest [cm]
$\bar{y}_i$, $\bar{z}_i$	Centroidal coordinates of the i^{th} material region relative to the modulus-weighted axes [cm]
$\bar{y}'_i$, $\bar{z}'_i$	Centroidal coordinates of the i^{th} material region relative to the $\hat{y}'$-$\hat{z}'$ axes [cm]
δ_a, δ_y, δ_z	Displacement components of a point located at the MW centroid [cm]
υ	Poisson's ratio [1]
ρ	Weight density of composite [g/cm^3]
ρ_i	Weight density of i^{th} material region [g/cm^3]
$(\sigma_o)_i$	Tensile yield strength of i^{th} material region [Pa]

τ_i	Torsional stress in circular annular cross-section of i^{th} material region [Pa]
$\tau_{\max}$	Maximum torsional stress in rectangular solid component [Pa]
τ_{xy} , τ_{xz}	Transverse shear stress components [Pa]
$(\tau_o)_i$	Shear yield strength of i^{th} material region [Pa]
$(\tau_i)_{b,h}$	Torsional stress in rectangular cross-section of i^{th} material region [Pa]
θ_p	Orientation of the principal axes relative to the $\hat{y}$-$\hat{z}$ axes [deg]
θ_x , θ_y , θ_z	Rotation components of a point located at the MW centroid [deg]
$(\bullet)_s$	Symmetric component of the quantity in parentheses
$(\bullet)_a$	Antisymmetric component of the quantity in parentheses